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Extremal properties of the unit ball in H^1

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ABSTRACT

Strongly exposed points of the unit ball in $H^1(D)$ are studied. We give criteria for extreme points to be strongly exposed. The main result is that an exposed point of the unit ball of H^1 is strongly exposed, if the L^∞-distance of $\bar{f}/|f|$ to $C(T)+H^\infty$ is less than 1.

1. INTRODUCTION

Let $B(H^1)$ denote the unit ball of the usual Hardy space H^1 on the unit disc $D\subset\mathbb{C}$ and let $\partial B(H^1)$ denote its boundary. We are interested in function theoretic interpretations of certain geometric properties of boundary points of $B(H^1)$. These properties are: to be an exposed or strongly exposed boundary point. Recall that if K is a convex set in a Banach space $\mathscr{B}$, a point $x\in K$ is called extreme if $K\setminus\{x\}$ is convex; a point x is called exposed in K if there exists $L\in\mathscr{B}^*$ such that $\mathrm{Re}\langle L,x\rangle>\mathrm{Re}\langle L,y\rangle$ for every $y\in K\setminus\{x\}$. If K is the unit ball in $\mathscr{B}$ this is equivalent to the existence of $L\in\mathscr{B}^*$ with

$$\langle L,x\rangle=\|L\|,$$

and for $y\in K$

$$\langle L,y\rangle=\|L\| \Rightarrow x=y.$$

The functional L is called an exposing functional for x. Finally, x is called strongly exposed if x is exposed (with exposing functional L) and has the additional property that if $\{x_n\}$ is a sequence such that

$$\mathrm{Re}\langle L, x_n\rangle \to \mathrm{Re}\langle L, x\rangle, \quad n\to\infty,$$

then

$$\|x-x_n\| \to 0, \quad n\to\infty.$$

Of course we have

strongly exposed $\Rightarrow$ exposed $\Rightarrow$ extreme.

An exposed point that is not strongly exposed will be called weakly exposed. An example of a weakly exposed point of a convex set is the point $(0,0,\ldots)\in \{x\in l^2\colon x_i\geq 0\}$ with exposing functional $L=(-1,-1/2,-1/3,\ldots)$.

One motivation for studying strongly exposed points are refinements of the Krein Milman theorem, due to Phelps [Ph], see also Kunen–Rosental and [KR]. Since we are only interested here in H^1, we formulate a special case of these results, which fits H^1.

THEOREM (Phelps). *Let K denote a closed bounded convex set in a dual Banach space, then K is the norm closure of the convex hull of the strongly exposed points of K.*

Now we turn to $B(H^1)$. It is a result of deLeeuw and Rudin [dLR] that f is an extreme point of $B(H^1)$ if and only if $\|f\|_1=1$ and f is an outer function. There is no such satisfying description of exposed points, but there are positive results, cf. [H], [Na85], [S89], [S90], [Y72]. E.g. a complex polynomial $p\in \partial B(H^1)$ is exposed if and only if all its zeros are outside the open unit disc, and it has only zeros of order 1 on the unit circle T.

Section 2 is about exposed points. We review some results that are of interest for the next section and give a simple sufficient condition for exposedness related to our main result.

In Section 3 and 4 we deal with strongly exposed points. Our main result is as follows:

THEOREM. *If f is exposed in $B(H^1)$ and $d(\bar{f}/|f|, C(T)+H^\infty)<1$, then f is strongly exposed.*

Here d is the L^∞ distance. Note that in general $d(\bar{f}/|f|, C(T)+H^\infty)\leq 1$. We present two proofs. One method is "direct" and gives also a sufficient condition for $f\in L^\infty$ to admit a dual extremal function $F\in H_0^1=zH^1$ measuring the distance $d(f,H^\infty)$. The other, in Section 4, is based on Banach algebra ideas and is much shorter. While finishing this paper, we became aware of [Na91]. Here a Banach algebra proof of the result on existence of dual extremal functions is given. It is closely related to our proof in Section 4.

In Section 5 we shall give examples of exposed points that are not strongly exposed.

A lot of information on exposed points is available. Given a point $f \in \partial B(H^1)$, the exposing functional is unique up to a multiplicative constant and may be represented by integration against the L^∞-function $\bar{f}/|f|$. Thus, a function $f \in \partial B(H^1)$ is exposed if and only if

(2.1) $\quad \arg g = \arg f$ a.e. on T for some $g \in H^1 \Rightarrow g = cf, \quad c>0,$

cf. [dLR]. This can serve as an alternative definition. DeLeeuw and Rudin also found the following sufficient condition: for $f \in \partial B(H^1)$

(2.2) $\quad$ 1/f?H$^\infty$ $\Rightarrow f$ is exposed point of $B(H^1)$.

K. Yabuta [Y71] strengthened this by proving

(2.3) $\quad 1/f \in H^1 \Rightarrow f$ is exposed.

From other work of Yabuta [Y72] it follows that for $f \in \partial B(H^1)$,

(2.4) $\quad \exists\, k \in H^\infty$ such that $\operatorname{Re} kf \geq 0$ a.e. on $T \Rightarrow f$ is exposed.

While Yabuta considered the Hardy class on the unit polydisc, we will confine ourselves to the one-dimensional disc.

In 1985 Nakazi [Na85] showed that a function $f \in H^1$ can be approximated in the L^1-topology by functions $k \in H^1$ with $\arg k = \arg f$ a.e. on T and $k(z)/((z-a)(1-\bar{a}z)) \in H^1$ for some $a \in \bar{D}$, unless f is exposed.

Another point of view was taken by Hayashi [H]. He gave a characterization of the exposed points of $B(H^1)$ in terms of Toeplitz-operators on H^2. Very important work in this direction is due to Sarason, [S89], [S90].

The results of Yabuta yield that a polynomial $p \in \partial B(H^1)$ is exposed if and only if p is zero free in D and has no zeros of order >1 on T. More generally, a function f that is analytic in a neighborhood of $\bar{D}$ with a zero on T of order >1 (or, equivalently, $\arg f$ makes a jump $>\pi$ on T) cannot be exposed. Inoue and Nakazi [IN] gave an alternative proof for the polynomials by approximating $\arg p$ with a Lipschitz-continuous function. We will use this idea in the proof of Theorem 2.1 to show exposedness for another class of functions.

THEOREM 2.1. *Let $f \in \partial B(H^1)$ be outer. If $d(\arg f, C(T)+H^\infty) < \pi/2$, then f is exposed.*

PROOF. Note that $\arg f$ is defined a.e. on T, because it is the harmonic conjugate of $\log|f|$, which is integrable. There exist $\varepsilon>0$, $h \in H^\infty$ and $g \in C^\infty(T)$ such that $\|\arg f - g - h\|_\infty < \pi/2 - \varepsilon$. Since $\arg f$ is real-valued, we have $\|\arg f - \operatorname{Re} g - \operatorname{Re} h\|_\infty < \pi/2 - \varepsilon$. Let H be the H^∞-function with $\operatorname{Re} H = \operatorname{Re} g + \operatorname{Re} h$ a.e. on T (it exists by elementary theory of harmonic conjugates). Then $\exp(-iH) \in H^\infty$ and $\operatorname{Re} \exp(-iH) f \geq 0$ a.e. on T. By (2.4) f is exposed. $\square$

Recently Nakazi [Na91] found two more abstract characterizations for exposed points.

We recall the following result of Newman.

THEOREM 3.1 [Ne]. *If $\{f_n\} \subset B(H^1)$ tends to f uniformly on compact sets, and $\|f_n\|_1 \to \|f\|_1$, then f_n tends to f strongly.*

The idea behind the proof of the next theorem is iteration of the proof of theorem 4.4, p. 153 in [G].

THEOREM 3.2. *If f is exposed in $B(H^1)$ and $d(\bar{f}/|f|, C(T)+H^\infty)<1$, then f is strongly exposed.*

PROOF. Put $h := \bar{f}/|f|$ and let $F_n \in B(H^1)$ be such that

$$(3.1) \qquad \frac{1}{2\pi}\int_0^{2\pi} F_n(e^{it})\, h(e^{it})\, dt \to 1, \quad n \to \infty.$$

We claim that every subsequence of $\{F_n\}$ has a weakly convergent subsequence. Assuming this for a moment, consider a subsequence $\{F_k\}$ converging weakly to some $F \in H^1$. Of course we also have $\|F\|_1 \le 1$. Furthermore

$$\frac{1}{2\pi}\int_0^{2\pi} F(e^{it})\, h(e^{it})\, dt = \lim_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} F_k(e^{it})\, h(e^{it})\, dt = 1$$

which gives that $F=f$, because f is exposed, thus all weak limit points coincide. It follows that F_n converges weakly, and then uniformly on compacta as well, to f. Now we can use the above theorem of Newman and conclude $\|F_n - f\|_1 \to 0$, therefore f is strongly exposed.

Now assume that the claim were false. Then we would have a subsequence $\{F_j\}$ without weakly converging subsequence, and there are sets $E_k \subset T$ such that $|E_k| \to 0$, while

$$(3.2) \qquad \Big|\int_{E_k} F_k\Big| \ge \beta > 0$$

for a subsequence $\{F_k\}$ of $\{F_j\}$ (see [DS], p. 292). Next we use a lemma from [G], p. 153:

LEMMA 3.3. *If $\{E_k\}$ is a sequence of measurable subsets of T, such that $|E_k| \to 0$, then there are a sequence $\{g_k\} \subset H^\infty$ and $\varepsilon_k \downarrow 0$, such that*

(i) $\sup_{E_k} |g_k| \to 0$,
(ii) $g_k(0) \to 1$,
(iii) $|g_k| + |1-g_k| \le 1+\varepsilon_k$.

Continuing with the proof of Theorem 3.2, let

$$G_{1,k} := \frac{g_k F_k}{1+\varepsilon_k} \quad \text{and} \quad H_{1,k} := \frac{(1-g_k)F_k}{1+\varepsilon_k}.$$

Then $G_{1,k}, H_{1,k} \in H^1$. Next we define for $n>1$

$$G_{n,k} := \frac{g_k H_{n-1,k}}{(1+\varepsilon_k)\|H_{n-1,k}\|}, \qquad H_{n,k} := \frac{(1-g_k) H_{n-1,k}}{(1+\varepsilon_k)\|H_{n-1,k}\|}.$$

We shall show that for every n we have the following:

(a) $\|G_{n,k}\|_1 + \|H_{n,k}\|_1 \leq 1$.

(b) For k large enough, $\|H_{n,k}\|_1 \geq \beta/2$.

(c) $$\|G_{n,k}\|_1 \int h \frac{G_{n,k}}{\|G_{n,k}\|_1} \frac{d\theta}{2\pi} + \|H_{n,k}\|_1 \int h \frac{H_{n,k}}{\|H_{n,k}\|_1} \frac{d\theta}{2\pi} \to 1 \quad \text{for } k \to \infty.$$

(d) $\lim_{k\to\infty} \int hH_{n,k}/\|H_{n,k}\|_1 = 1$.

(e) For $0 \leq i < n$ the i-th derivative of $H_{n,k}$ satisfies $\lim_{k\to\infty} H_{n,k}^{(i)}(0) = 0$.

Property (a) is true by Lemma 3.3(iii) for all $n \geq 1$. For $n=1$, (b) follows from (3.2) and (i) of the lemma and (c) from (3.1). Combining (a), (b) and (c) then gives (d) and finally (e) is immediate from Lemma 3.3(ii).

Next take $n>1$. From (a) we have $\|H_{n,k}\|_1 \leq 1$ for all k. Using this with (3.2) and Lemma 3.3(i) we prove (b):

$$\|H_{n,k}\|_1 = \left\| \frac{(1-g_k)^n F_k}{(1+\varepsilon_k)^n \prod_{j=1}^{n-1} \|H_{j,k}\|} \right\|_1 \geq \frac{\|(1-g_k)^n F_k\|_1}{|1+\varepsilon_k|^n}$$

$$\geq \frac{1}{|1+\varepsilon_k|^n} \int_{E_k} |1-g_k|^n |F_k| > \beta/2,$$

for k large enough. Property (c) reduces to the $n-1$ case of (d) and (d) follows, again, from (a), (b) and (c). To prove (e), we have, for $0 \leq i < n$,

$$H_{n,k}^{(i)}(0) = \sum_{j=0}^{i} \binom{i}{j} \frac{(1-g_k)^{(j)}(0)\, H_{n-1,k}^{(i-j)}(0)}{(1+\varepsilon_k)\|H_{n-1,k}\|_1}.$$

If $i<n-1$, the factors $(1-g_k)^{(j)}(0)$ are bounded for $k\to\infty$ (e.g. by the Cauchy-formula) and $H_{n-1,k}^{(i-j)}(0) \to 0$ by induction. The denominators are $>\beta/2>0$. If $i=n-1, j>0$ then the terms tend to 0 for the same reason, and for $i=n-1$ and $j=0$ the values $H_{n-1,k}^{(i-j)}(0)$ are bounded and $(1-g_k)^{(j)}(0) \to 0$ by (ii) of the lemma. So all terms tend to 0 and we have (e).

Taking $H_n := H_{n,k_n}/\|H_{n,k_n}\|_1$ with k_n large enough, we obtain a sequence in $B(H^1)$ which by (d) and (e) has the properties $\int hH_n \to 1$ and form $m \geq 0$, $\int e^{-imt} H_n(e^{it})\,dt \to 0$. For $g \in C(T)+H^\infty = C(T)+H_0^\infty$, using that continuous functions can be approximated by trigonometric polynomials and that $(H^1)^\perp = H_0^\infty$, we can conclude that $\int H_n g \to 0$ for $n\to\infty$.

By assumption there is a $g \in C(T)+H^\infty$ with $\|h-g\|_\infty < 1$. With this g we get

$$1 = \lim_{n\to\infty} \left|\int H_n h\right| = \lim_{n\to\infty} \left|\int H_n(h-g)\right| \leq \lim_{n\to\infty} \|H_n\|_1 \|h-g\|_\infty < 1,$$

which is a contradiction. Therefore $\{F_n\}$ is weakly convergent and f is strongly exposed. □

The following corollaries may be compared to (2.4) and (2.3).

COROLLARY 3.4. *Let $f\in\partial B(H^1)$ be outer. If $d(\arg f, C(T)+H^\infty)<\pi/2$, then f is strongly exposed.*

PROOF. By Theorem 2.1 f is exposed. Note that $d(\arg f, C(T)+H^\infty)<\pi/2$ implies that $d(\arg f, \operatorname{Re} H^\infty)<\pi/2$. Then it is easy to see, and it also follows from Lemma 4.3.3 in [G], p. 148 that $d(\bar{f}/|f|, C(T)+H^\infty)<1$. By Theorem 3.2 f is strongly exposed. □

E.g., we obtain that for $-1<\alpha<1$ the functions $(z-1)^\alpha/\|(z-1)^\alpha\|_1$ are strongly exposed.

COROLLARY 3.5. *If $f\in\partial B(H^1)$ and $\exists\,\varepsilon>0$ such that $|\arg f|<\pi/2-\varepsilon$ in D, then f is strongly exposed.*

PROOF. Since $\operatorname{Re} f\geq 0$, f is outer and 3.4 applies. □

COROLLARY 3.6. *If $f\in H^\infty\subset H^1$, $1/f\in H^\infty$ and $\|f\|_1=1$, then f is strongly exposed.*

PROOF. Let $\delta<\|f\|_\infty<M$. Note that

$$\left\|\frac{\bar{f}}{|f|}-\frac{\delta}{f}\right\|_\infty=\left\|\frac{|f|-\delta}{|f|}\right\|_\infty<1-\frac{\delta}{M}$$

and apply (2.2) and Theorem 3.2. □

The same technique as in the proof of Theorem 3.2 can be used to prove the following result. In [Na91] a Banach Algebra proof was given.

THEOREM 3.7. *Let $f\in L^\infty$. If $d(f, C(T)+H^\infty)<d(f,H^\infty)$, then there is a function $F\in H_0^1$, $\|F\|_1=1$, such that $\int fF=d(f,H^\infty)$.*

PROOF. Because $d(f,H^\infty)=\sup\{|\int fF|: F\in H_0^1,\ \|F\|_1\leq 1\}$ we can take $F_n\in B(H_0^1)$ such that $\int fF_n\to d(f,H^\infty)$. Every weak limit of a subsequence of $\{F_n\}$ makes the statement true.

Assume $\{F_n\}$ has no weakly convergent subsequence, then, like in the proof of Theorem 3.2, we can construct functions $H_n\in B(H_0^1)$, $\|H_n\|_1=1$, such that $\int fH_n\to d(f,H^\infty)$ and $\int gH_n\to 0$ for all $g\in C(T)$. Taking $h+g\in H^\infty+C(T)$ such that $\|f-(h+g)\|_\infty<d(f,H^\infty)$ we find the contradiction

$$\begin{aligned}d(f,H^\infty)&=\lim_{n\to\infty}\int fH_n=\lim_{n\to\infty}\int(f-(g+h))H_n\\&\leq\lim_{n\to\infty}\|f-(g+h)\|_\infty\|H_n\|_1<d(f,H^\infty).\end{aligned}$$

Therefore, there exists a subsequence of $\{F_n\}$ converging weakly to, say, $F \in B(H_0^1)$. For this F we have $\int fF = d(f, H^\infty)$. $\square$

4. A BANACH ALGEBRA APPROACH

We found a Banach algebra proof of Theorem 3.2 which is more abstract, but in our opinion it is also more motivated. It is based on the Helson–Lowdenslager generalization of the F.&M. Riesz theorem. We refer to [G, Ho] for the necessary background in Uniform Algebras.

Let $\mathcal{M}$ denote the maximal ideal space of H^∞, and X its Shilov boundary. The maximal ideal space of L^∞ coincides with X. Recall that H^∞ is a log-modular algebra on X, so that every homomorphism has a unique representing measure on X. Let μ_0 denote the unique representing measure for point evaluation at 0. As usual we denote Gelfand transform with $\hat{}$. For a measurable set $E \subset T$, the characteristic function χ_E is an idempotent in L^∞. Its Gelfand transform $\hat{\chi}_E$ assumes only the values 0 and 1 on X and is the characteristic function of a set $\tilde{E} \subset X$. Note that

$$\mu_0(\tilde{E}) = \int_X \chi_{\tilde{E}}\, d\mu_0 = \int_T \chi_E\, d\theta/2\pi.$$

This completely determines μ_0. Moreover the following isometry theorem is known, cf. [G], p. 202:

THEOREM 4.1. *Let $0 < p \leq \infty$. Then the correspondence*

$$\chi_E \to \chi_{\tilde{E}}$$

extends to a unique positive isometric linear operator from $L^p(T, d\theta/2\pi)$ onto $L^p(X, \mu_0)$.

Now we formulate the F.&M. Riesz theorem we need, see [G, Ho]:

THEOREM 4.2. *Let A be a uniform algebra on a compact Hausdorff space Y and let m be a homomorphism with unique representing measure μ on Y. If $\nu \in A_m^\perp$ has Lebesgue decomposition $\nu = \nu_a + \nu_s$ with respect to μ, then $\nu_a \in A_m^\perp$ and $\nu_s \in A^\perp$.*

Here A_m is the ideal corresponding to m and $A^\perp$ denotes the space of Borel measures orthogonal to A, while $A_m^\perp$ is defined similarly.

ALTERNATIVE PROOF OF THEOREM 3.2. The functions $F_n \in H^1$ are viewed as functionals, first on $C(T)$ and next on L^∞, which we identify via the Gelfand transform with $C(X)$. On $C(T)$ we find a subsequence F_{n_j} converging weak* to a measure μ on T which again is an H^1 function, hence a functional on L^∞. Let $\hat{F}_{n_j}$ and $\hat{\mu}$ denote the measures on X which represent the functionals F_{n_j} and μ respectively. On X the sequence $\hat{F}_{n_j}$ has a subsequence converging weak* to a measure ν on X. Note that $\|\nu\| = 1$ $(= \int \hat{h}\, d\nu)$, while $\|\mu\| \leq 1$. We wish to show that $\hat{\mu} \perp \nu - \hat{\mu}$. By the isometry theorem $\hat{\mu} \ll \mu_0$. Observe that $\nu - \hat{\mu}$ annihilates $\widehat{H^\infty + C(T)}$. In other words, for every $m \in \mathbb{Z}$ we have $\widehat{e^{imt}}(\nu - \hat{\mu}) \perp \widehat{H^\infty}$. We use the modern F.&M. Riesz theorem and the isometry theorem to obtain

$$\widehat{e^{imt}}(\nu - \hat{\mu}) = \hat{h}_m \mu_0 + \nu_s^m,$$

with $h_m \in L^1(T, d\theta/2\pi)$. In view of the uniqueness of the Lebesgue decomposition, we see that $h_m = e^{imt} h_0$ and the orthogonality property of $\hat{h}_m \mu_0$ gives

$$\int h_0 e^{im\theta}\, d\theta = 0, \quad m \in \mathbb{Z}.$$

It follows that $h_0 = 0$, and $\hat{\mu} \perp \nu - \hat{\mu}$, so that $1 = \|\nu\| = \|\hat{\mu}\| + \|\nu - \hat{\mu}\|$. We have

$$1 = \int \hat{h}\, d\nu = \int \hat{h}\, d\hat{\mu} + \int \hat{h}\, d(\nu - \hat{\mu}) \leq \left|\int \hat{h}\, d\hat{\mu}\right| + d(h, H^\infty + C(T))\, \|\nu - \hat{\mu}\|,$$

since $\nu - \hat{\mu}$ annihilates $\widehat{H^\infty + C(T)}$. Now $d(h, H^\infty + C(T)) < 1$ implies that $\nu = \hat{\mu}$. Because f is exposed, the H^1 function μ equals f independently of the choice of the subsequence F_{n_j}. We find that the sequence F_n converges weak to f. By Newman's Theorem we are done. □

5. EXAMPLES

Theorem 3.2 does not give necessary conditions for strong exposedness. In this section we give examples of weakly exposed points of $B(H^1)$.

EXAMPLE 5.1. A complex polynomial, zero free in the disc but with at least one single zero in the circle, is not strongly exposed. We give the following example, but it is clear that the same method works for general polynomials, or even arbitrary exposed functions f with the property that for some $\zeta \in T$ the function $f/(z-\zeta)^{2-\varepsilon}$ is in $B(H^1)$ for all $\varepsilon > 0$. Indeed, taking

$$f(z) = \frac{z+1}{\|z+1\|_1} \quad \text{and} \quad f_n(z) = c_n (z+1)^{1+1/n} \left(-\left(\frac{z-1}{z+1}\right)^2\right),$$

with constants $c_n > 0$ such that $\|f_n\|_1 = 1$, we have a sequence in $B(H^1)$ with

$$\begin{aligned}\int f_n \frac{\bar{f}}{|f|} &= \int |f_n|\, e^{i \arg f_n} e^{-i \arg f} \\ &= \int |f_n|\, e^{(1+1/n) i \arg f} e^{-i \arg f} = \int |f_n|\, e^{(1/n) i \arg f} \to \int |f_n| = 1,\end{aligned}$$

while $f_n \not\to f$ in the norm-topology. The functions f_n are tending pointwise to 0 in D, for the constants c_n are decreasing to 0 in order to keep f_n in $B(H^1)$.

EXAMPLE 5.2. A small variation of 5.1 will give an example of an exposed point f of $B(H^1)$ with the property that $1/f \in H^1$ such that f is not strongly exposed, cf. (2.3).

Let $f = c(z-1)(\log(z-1))^2$, where $c > 0$ is such that $\|f\|_1 = 1$. Then it is easily seen that $1/f \in H^1$, but as in the previous example f is not strongly exposed, because the sequence $f_n(z) = -c_n((z+1)/(z-1))^2 f^{(1+1/n)}(z)$, where $c_n > 0$ is such that $\|f_n\|_1 = 1$, has the property that f_n tends to 0 u.c., but $\int f_n(\bar{f}/|f|)$ tends to 1.

EXAMPLE 5.3. There exists a strongly exposed point f such that $d(\bar{f}/|f|, C(T)) = 1$. Let g be the conformal map from the unit disc to G, where

$$G = \{0<x<1, -10<y<5\sin(1/x)\}.$$

Then $\operatorname{Re} g$ is continuous, while $d(\operatorname{Im} gC(T)) = 5$, cf. [G], p. 377. Let $f(z) = e^{g(z)}$, then f and $1/f$ belong to H^∞, so by Corollary 3.6 $f/\|f\|_1$ is strongly exposed. However $d((\bar{f}/|f|), C(T)) = 1$.

REFERENCES

[DS] Dunford, N. and J. Schwartz - Linear operators, Part I. Wiley (Interscience), New York 1958.

[G] Garnett, J.B. - Bounded analytic functions. Academic Press, San Diego, New York etc. 1981.

[H] Hayashi, E. - The solution of extremal problems in H^1. Proc. Amer. Math. Soc. **93**, 690-696 (1985).

[Ho] Hoffman, K. - Banach spaces of analytic functions. Prentice Hall, Englewood Cliffs, N.J. 1962.

[IN] Inoue, J. and T. Nakazi - Polynomials of an inner function which are exposed points in H^1. Proc. Amer. Math. Soc. **100**, 454-456 (1987).

[KR] Kunen, K. and H.P. Rosenthal - Martingale proofs of some geometrical results in Banach spaces. Pacific J. Math. **100**, 153-175 (1982).

[dLR] Leeuw, K. de and W. Rudin - Exposed points and extremum problems in H^1. Pacific J. Math. **8**, 467-485 (1958).

[Na83] Nakazi, T. - Exposed points and extremal problems in H^1. J. Funct. Anal. **53**, 224-230 (1983).

[Na85] Nakazi, T. - Exposed points and extremal problems in H^1, II. Tôhoku Math. J. **37**, 265-269 (1985).

[Na91] Nakazi, T. - Existence of solutions of extremal problems in H^1. Proc. Edinburgh Math. Soc. **34**, 99-112 (1991).

[Ne] Newman, D.J. - Pseudo-uniform convexity in H^1. Proc. Amer. Math. Soc. **14**, 676-679 (1963).

[Ph] Phelps, R.R. - Dentability and extreme points in Banach spaces. J. Funct. Anal. **16**, 78-90 (1974).

[S89] Sarason, D.E. - Exposed points in H^1, I. In: Operator Theory: Advances and Applications **41**, Birkhauser Verlag Basel, 485-496 (1989).

[S90] Sarason, D.E. - Exposed points in H^1, II. In: Operator Theory: Advances and Applications **48**, Birkhauser Verlag Basel, 333-347 (1990).

[Y71] Yabuta, K. - Unicity of the extremum problems in $H^1(U^n)$. Proc. Amer. Math. Soc. **28**, 181-184 (1971).

[Y72] Yabuta, K. - Some uniqueness theorems for $H^p(U^n)$ functions. Tôhoku Math. J. **24**, 353-357 (1972).